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Extended state model and group approach to new polynomials

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Received 17 April 1989, in final form 30 August 1989

Abstract. The extended Kauffman state model is presented by introducing the annihilation diagrams. This new state model enables us to establish a connection between the Akutsu-Wadati type polynomials and the group approach of Witten.

1. Introduction

In an elegant survey by Kauffman [1] the knot diagram theory was discussed through the state models, i.e. bracket polynomial, surface state and S -matrix model [2]. The diagrammatic version has been shown to be a powerful method of constructing the polynomials, including the Jones polynomial, which are invariant under the Reidemeister moves of type I, II and III. Among them the S -matrix state model which specifies elements of a representation of the braid group is a direct model and is closely related to the S -matrix with infinite rapidity in physics.

In this paper we shall present an extended S -matrix model which leads to the Akutsu-Wadati (AW) type polynomials and show that this approach provides a graphical description of the projection to a plane of a knot (link) in 3-space for the polynomial theory of Witten [3, 4].

Let us begin with the Temperley-Lieb representation of the braid group [4, 5]

$$T(u) = u \sum E_{aa} \times E_{aa} + \sum_{a \neq b} E_{ab} \times E_{ba} - (u - u^{-1}) \sum_{a < b} E_{aa} \times E_{bb} \quad (1.1)$$

where u is a parameter and E signifies the unit matrix. In (1.1) the indices a, b, c and d label the spin. Following Kauffman, (1.1) can be represented by the diagrammatic expansion

$$T_{cd}^{ab} = \begin{array}{c} \diagdown \\ \diagup \end{array} = u \begin{array}{c} a \quad b \\ \downarrow \quad \downarrow \\ \text{---} \\ \downarrow \quad \downarrow \\ c \quad d \end{array} + \begin{array}{c} a \neq b \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} + (u - u^{-1}) \begin{array}{c} a < b \\ \downarrow \quad \downarrow \\ \text{---} \\ \downarrow \quad \downarrow \\ c \quad d \end{array} \quad (1.2)$$

where the notation is the same as that in [1, 2].

Obviously T_{cd}^{ab} introduced here represents an element of the S -matrix with the infinite rapidity, this is because the S -matrix is related to the R -matrix through transfer matrix theory, but the limit of the S -matrix at infinite rapidity satisfies the Yang-Baxter equations [6, 7]. Henceforward we denote by T the S -matrix with infinite rapidity, i.e. a representation of the braid group (BGR).

In (1.9) $[K|S]$ denotes a product of local contributions from the vertices of the diagrams. For a state S , $\|S\|$ is defined as

$$\|S\| = \sum_{l \in \text{Comp}(S)} \text{rot}(l) \text{label}(l) \tag{1.10}$$

where $\text{comp}(S)$ denotes the components of S , $\text{label}(l)$ is the spin assigned to l and $\text{rot}(l) = 1$ or -1 according to whether the loop is anticlockwise or clockwise. $[\bigcirc]$ in (1.8) represents unknotted $[K]$. In (1.8) $\tilde{\alpha}$ means

$$\left[\text{loop with arrow} \right] = \tilde{\alpha} \left[\text{loop} \right] \tag{1.11}$$

and t will be determined by cross-channel unitarity [2]. Since the ‘adjustment factor’ $t^{\|S\|}$ appears only in a trace (summation over all the states), it is enough to use ‘trace cross-channel unitarity’ to determine the relationship between t in (1.9) and the parameter u in a representation of the braid group, for example, T_{cd}^{ab} in (1.2) [2, 7].

The trace cross-channel unitarity can be expressed by

$$\left[\text{figure-eight with arrows} \right] = \left[\text{loop with arrow} \right] \tag{1.12}$$

which leads to algebraic equations determining $t = t(u)$. It is easy to see that (1.12) is the only possibility for realising the cross-channel unitarity in a trace [2].

For irreducible representations of SL_2 , AW have discovered the new polynomials and skein relations based on the six-vertex models [7]. In their three examples the reduction relations (skein relations) are ‘higher’ than Jones’ in the power of the braid group generator for spin $\frac{1}{2}$, 1 and $\frac{3}{2}$, or equivalently, $N = n + 1 = 2s + 1 = 2, 3$ and 4.

Recently, Witten gave a natural framework for understanding the Jones polynomial of knot theory in $(2 + 1)$ -dimensional quantum Yang–Mills theory. For the fundamental representations of $SU(N)$ the conformal weights of a primary conformal field were introduced to calculate the eigenvalues of the matrix B in the convention of Moore and Seiberg [9].

In this paper we shall first extend Kauffman’s diagrammatic scheme to calculate new polynomials and new skein relations including AW results. As an illustrative example, the calculations for $N = 3$ will be made with the help of the extended diagrammatic technique. In the same way the calculations are also made for $N = 4$ and 5.

We shall extend Witten’s discussions to give a general scheme for calculating AW type polynomials for any spin.

The above coincidence suggests that the diagrammatic scheme depends on the group. If we are able to extend our approach to the groups B_n , C_n and D_n , an infinite number of new reduction relations, therefore an infinite number of polynomials, can be expected. The answer is yes. We have developed the idea of this paper to give the relevant calculations for fundamental representations of B_n , C_n and D_n . The calculations will be published elsewhere. Thus this paper, in a sense, is an introduction to a great number of new reduction relations and polynomials far beyond those of Jones, Kauffman and AW, although the Jones polynomial is the ‘co-starting point’.

2. The extended diagrammatic calculations for spin 1, $\frac{3}{2}$ and 2

In this section we calculate the polynomials and skein relations for spin 1 and $\frac{3}{2}$ ($N = 3$ and 4, respectively) in terms of the generalised diagrammatic technique. Follow Kauffman [1, 2], the S -matrix can be defined as

$$T_{cd}^{ab} = \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \quad d \end{array} \quad (T^{-1})_{cd}^{ab} = \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ c \quad d \end{array} \tag{2.1}$$

2.1. New polynomial for $N = 3$

Considering zero elements in the upper-left triangle of the submatrices A_m in (1.7), we introduce

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \quad d \end{array} = u \begin{array}{c} a \quad a \\ \diagdown \quad \diagup \\ a \quad a \end{array} + w \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ a \quad b \end{array} + p \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ b \quad a \end{array} + v' \left(\begin{array}{c} 0 \quad 0 \\ \diagdown \quad \diagup \\ -2 \quad 2 \end{array} + \begin{array}{c} -2 \quad 2 \\ \diagup \quad \diagdown \\ 0 \quad 0 \end{array} \right) \tag{2.2}$$

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \quad d \end{array} = u' \begin{array}{c} a \quad a \\ \diagdown \quad \diagup \\ a \quad a \end{array} + w' \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ a \quad b \end{array} + p' \begin{array}{c} a \quad b \\ \diagup \quad \diagdown \\ b \quad a \end{array} + v \left(\begin{array}{c} 0 \quad 0 \\ \diagdown \quad \diagup \\ 2 \quad -2 \end{array} + \begin{array}{c} 2 \quad -2 \\ \diagup \quad \diagdown \\ 0 \quad 0 \end{array} \right) \tag{2.3}$$

where the spin indices a, b, c and $d \in (-2, 0, 2)$ and

$$\begin{aligned} u &= u_1(\delta_{a_2} + \delta_{a_{-2}}) + u_2\delta_{a_0}, & w &= w_1(\delta_{a_0}\delta_{b_2} + \delta_{a_{-2}}\delta_{b_0}) + w_2\delta_{a_{-2}}\delta_{b_2} \\ p &= p_1(\delta_{a_0}\delta_{b_{-2}} + \delta_{a_{-2}}\delta_{b_0} + \delta_{a_2}\delta_{b_0} + \delta_{a_0}\delta_{b_2}) + p_2(\delta_{a_2}\delta_{b_{-2}} + \delta_{a_{-2}}\delta_{b_2}) \\ v &= v_1(\delta_{a_0}\delta_{b_0}\delta_{c_{-2}}\delta_{d_2} + \delta_{a_{-2}}\delta_{b_2}\delta_{c_0}\delta_{d_0}) \\ u' &= u'_1(\delta_{a_2} + \delta_{a_{-2}}) + u'_2\delta_{a_0} & w' &= w'_1(\delta_{a_2}\delta_{b_0} + \delta_{a_0}\delta_{b_{-2}}) + w'_2\delta_{a_2}\delta_{b_{-2}} \\ p' &= p'_1(\delta_{a_0}\delta_{b_{-2}} + \delta_{a_{-2}}\delta_{b_0} + \delta_{a_2}\delta_{b_0} + \delta_{a_0}\delta_{b_2}) + p'_2(\delta_{a_2}\delta_{b_{-2}} + \delta_{a_{-2}}\delta_{b_2}) \\ v' &= v'_1(\delta_{a_0}\delta_{b_0}\delta_{c_2}\delta_{d_{-2}} + \delta_{a_2}\delta_{b_{-2}}\delta_{c_0}\delta_{d_0}) \end{aligned} \tag{2.4}$$

i.e. in (2.1) the ‘annihilation terms’ are introduced to extend the original diagrams with only the ‘scattering terms’. Now we require (2.2) and (2.3) to satisfy (1.3), which gives

$$\begin{aligned} u_1 u'_1 = u_2 u'_2 = p_1 p'_1 = p_2 p'_2 = 1 & \quad w_1 p'_1 + p_1 w'_1 = 0 \\ w_2 p'_2 + p_2 w'_2 + v_1 v'_1 = 0 & \quad u_2 v'_1 + v_1 p'_2 = 0. \end{aligned} \tag{2.5}$$

The solution reads

$$\begin{aligned} u'_1 = u_1^{-1} & \quad u'_2 = u_2^{-1} & p'_1 = p_1^{-1} & \quad p'_2 = p_2^{-1} \\ v'_1 = -v_1/u_2 p_2 & \quad w'_1 = -w_1/p_1^2 & w'_2 = (v_1^2 - u_2 w_2)/p_2^2. \end{aligned} \tag{2.6}$$

By using the symbolic expansions of the extended state model (2.2), the ΥB equations (1.4) can be split into states in (2.2) and (2.3) that provide the algebraic equations for constraining the parameters appearing in (2.2) and (2.3).

For $a = b = d = c, c = f, a \neq c$ we have

$$u^2 w \begin{array}{c} a \quad a \quad c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + u_1 v_1^2 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ 0 \quad 2 \quad 2 \end{array} = u w^2 \begin{array}{c} a \quad a \quad c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + w p^2 \begin{array}{c} a \quad a \quad c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + w_1 v_1^2 \begin{array}{c} -2 \quad -2 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad -2 \quad 2 \end{array}.$$

For $a = d, b = e, c = f, a \neq b \neq c$ we have

$$w^3 \begin{array}{c} a < b < c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + w \begin{array}{c} a \quad b \quad c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + u_2 v_1^2 \begin{array}{c} -2 \quad 2 \quad 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 0 \end{array} \\ = w^3 \begin{array}{c} a < b < c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + w p^2 \begin{array}{c} a \quad b \quad c \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + u_2 v_1^2 \begin{array}{c} 0 \quad -2 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ 0 \quad -2 \quad 2 \end{array}.$$

For $a = b = 0, c = 2$ we have

$$u_1 w_2 v_1 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + u_1 p_2 v_1 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ 2 \quad -2 \quad 2 \end{array} + w_1 u_2 v_1 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} \\ = u_1 w_1 v_1 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad 2 \quad 2 \end{array} + p_1^2 v_1 \begin{array}{c} 0 \quad 0 \quad 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ 2 \quad -2 \quad 2 \end{array}.$$

For $a = c = 0, b = -2$ we have

$$u_1 p_1 v_1 \begin{array}{c} 0 \quad -2 \quad 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad -2 \quad 2 \end{array} = p_1 p_2 v_1 \begin{array}{c} 0 \quad -2 \quad 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad -2 \quad 2 \end{array} + p_1 w_1 v_1 \begin{array}{c} 0 \quad -2 \quad 0 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \downarrow \downarrow \downarrow \\ -2 \quad -2 \quad 2 \end{array}.$$

The other equations do not give new results. The diagrams give the equations as follows:

$$\begin{aligned} u_1^2 w_1 &= u_1 w_1^2 + w_1 p_1^2 & u_2^2 w_2 + v^2 u_1 &= u_2 w_1^2 + w_1 p_1^2 \\ u_1^2 w_2 &= u_1 w_2^2 + w_2 p_2^2 + w_1 v_1^2 & w_1 p_2^2 + u_2 v_1^2 &= w_1 p_1^2 \\ u_1 w_2 + u_2 w_1 &= u_1 w_1 & u_1 p_2 &= p_1^2 & p_1^2 &= u_2^2 \end{aligned} \tag{2.7}$$

which has the solution set

$$\begin{aligned} p_2 &= u_1^{-1} p_1^2 & u_2 &= -p_1 & w_1 &= u_1 - u_1^{-1} p_1^2 \\ w_2 &= u_1^{-1} w_1 (u_1 - u_2) & v_1^2 &= u_2^{-1} w_1 (p_1^2 - p_2^2). \end{aligned} \tag{2.8}$$

The trace cross-channel unitarity in this case is

$$\begin{aligned}
 \text{Diagram} &= \text{Diagram}_1 + u_1 w'_1 \text{Diagram}_2 + u_1 w'_2 \text{Diagram}_3 + u_2 w'_1 \text{Diagram}_4 + u_1^{-1} w_1 \text{Diagram}_5 \\
 &+ u_1^{-1} w_2 \text{Diagram}_6 + u_2^{-1} w_1 \text{Diagram}_7 + w_1 w'_1 \text{Diagram}_8
 \end{aligned} \tag{2.9}$$

i.e.

$$u_1 w'_1 + u_1 w'_2 t^{-2} + u_2 w'_1 t^{-2} + u_1^{-1} w_1 + u_1^{-1} w_2 t^2 + u_2^{-1} w_1 t^2 + w_1 w'_1 = 0. \tag{2.10}$$

It follows from (2.6), (2.8) and (2.10) that for spin set $(-n, -n-2, \dots, n)$ we have (in this example $N = 3$)

$$\begin{aligned}
 u_1 &= t^2 & p_1 &= -1 & u_2 &= 1 & p_2 &= -t^{-2} & w_1 &= t^2 - t^{-2} \\
 w_2 &= t^2 - 1 - t^{-2} + t^{-4} & v_1 &= t(1 - t^{-4}) \\
 w'_1 &= t^{-2} - t^2 & w'_2 &= t^4 - t^2 - 1 - t^{-2} & v'_1 &= -t^3(1 - t^{-4}) \\
 u'_1 &= t^{-2} & u'_2 &= 1 & p'_1 &= -1 & p'_2 &= t^2.
 \end{aligned} \tag{2.11}$$

The skein relation can be expressed by

$$\begin{aligned}
 \text{Crossing} &= u_1'^2 \text{Diagram}_1 + (u_2'^2 + v_1'^2) \text{Diagram}_2 + (w_1'^2 - p_1'^2) \text{Diagram}_3 + (w_2'^2 - p_2'^2 - v_1'^2) \text{Diagram}_4 \\
 &+ w_1' p_1' \text{Diagram}_5 + w_2' p_2' \text{Diagram}_6 + p_1'^2 \text{Diagram}_7 + p_2'^2 \text{Diagram}_8 + (u_2' + w_2') v_1' \\
 &\times \left(\text{Diagram}_9 + \text{Diagram}_{10} \right) + p_2' v_1' \left(\text{Diagram}_{11} + \text{Diagram}_{12} \right).
 \end{aligned} \tag{2.12}$$

Using (2.12) the skein relation

$$\text{Crossing} + \alpha \text{Diagram}_9 + \beta \text{Diagram}_{10} = \gamma \text{Diagram}_{11} \tag{2.13}$$

can be solved; we have

$$\begin{aligned}
 \alpha &= -(t^4 - t^2 + t^{-2}) & \beta &= t^4 & \gamma &= t^6 - t^2 + 1 \\
 T^{-3} &= (t^4 - t^2 + t^{-2}) T^{-2} + (t^6 - t^2 + 1) T^{-1} - t^4.
 \end{aligned} \tag{2.14}$$

There is the similar skein relation for T with $t \rightarrow t^{-1}$ in (2.14). Introducing $t^2 = q$ and $g = qT^{-1}$, (2.14) can be written as

$$g^3 = (q^3 - q^2 + 1)g^2 + (q^5 - q^3 + q^2)g - q^5. \tag{2.15}$$

This is the AW reduction relation for spin = 1 [7, 8].

We next calculate the corresponding polynomial by (1.8). Because





$$\bigcirc = w_2 t^4 + 2u_1 + u_2 + 2w_1 t^2 = t^4(t^2 + 1 + t^{-2}) = t^4 \bigcirc = \hat{\alpha} \bigcirc \tag{2.16}$$

where the single loop is determined by (1.9) with the unknotted diagram

$$[\bigcirc] = t^n + t^{n-2} + \dots + t^n \tag{2.17}$$

it follows that

$$P_{2+} = q(1 - q^2 + q^3)P_+ + q^2(q^2 - q^3 + q^5)P_0 - q^8 P_- \tag{2.18}$$

where $P_{2+} = P$ , $P_+ = P$ , $P_- = P$ , $P_0 = P$  [7]. The representation of the braid group is easy to construct by using (1.5).

Equation (2.18) was derived by AW [7]. Checking the above process step by step, it is easy to find that the Kauffman state expression for the polynomial (1.8) is identical to the Markov trace in AW [7], if the extended state model (including the annihilation terms) is considered.

2.2. The case for $N = 4$

With the spin indices $l \in (-3, -1, 1, 3)$ the extended state expansion can be written in the form

$$\begin{aligned} \begin{array}{c} a \\ \diagdown \\ c \end{array} \begin{array}{c} b \\ \diagup \\ d \end{array} &= u \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} a \\ \diagup \\ a \end{array} + w \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} b \\ \diagup \\ b \end{array} + p \begin{array}{c} a \\ \diagdown \\ b \end{array} \begin{array}{c} b \\ \diagup \\ a \end{array} + v_1 \left(\begin{array}{c} 1 \\ \diagdown \\ -1 \end{array} \begin{array}{c} 1 \\ \diagup \\ 3 \end{array} + \begin{array}{c} -1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} -1 \\ \diagup \\ -3 \end{array} + \begin{array}{c} -1 \\ \diagdown \\ 1 \end{array} \begin{array}{c} -1 \\ \diagup \\ 1 \end{array} + \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} \begin{array}{c} -3 \\ \diagup \\ -1 \end{array} + \begin{array}{c} 3 \\ \diagdown \\ -1 \end{array} \begin{array}{c} 3 \\ \diagup \\ 1 \end{array} \right) \\ &+ v_2 \left(\begin{array}{c} 1 \\ \diagdown \\ -3 \end{array} \begin{array}{c} -1 \\ \diagup \\ 3 \end{array} + \begin{array}{c} -1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} -3 \\ \diagup \\ -1 \end{array} \right) + v_3 \left(\begin{array}{c} -1 \\ \diagdown \\ -3 \end{array} \begin{array}{c} 1 \\ \diagup \\ 3 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ -3 \end{array} \begin{array}{c} -3 \\ \diagup \\ 1 \end{array} \right) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \begin{array}{c} a \\ \diagdown \\ c \end{array} \begin{array}{c} b \\ \diagup \\ d \end{array} &= u \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} a \\ \diagup \\ a \end{array} + w' \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} b \\ \diagup \\ b \end{array} + p' \begin{array}{c} a \\ \diagdown \\ b \end{array} \begin{array}{c} b \\ \diagup \\ a \end{array} + v'_1 \left(\begin{array}{c} 1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} 1 \\ \diagup \\ -1 \end{array} + \begin{array}{c} -1 \\ \diagdown \\ 1 \end{array} \begin{array}{c} -1 \\ \diagup \\ -3 \end{array} + \begin{array}{c} 3 \\ \diagdown \\ 1 \end{array} \begin{array}{c} -1 \\ \diagup \\ 1 \end{array} + \begin{array}{c} 3 \\ \diagdown \\ -1 \end{array} \begin{array}{c} -3 \\ \diagup \\ -1 \end{array} \right) \\ &+ v'_2 \left(\begin{array}{c} -1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} 1 \\ \diagup \\ -3 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ -3 \end{array} \begin{array}{c} -3 \\ \diagup \\ 1 \end{array} \right) + v'_3 \left(\begin{array}{c} 1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} -1 \\ \diagup \\ -3 \end{array} + \begin{array}{c} -1 \\ \diagdown \\ 3 \end{array} \begin{array}{c} -3 \\ \diagup \\ 1 \end{array} \right). \end{aligned}$$

Follow the same strategy as $N = 3$, equations (1.3), (1.4) and (1.12) must be imposed on (2.19) which leads, after lengthy calculations, to

$$\begin{aligned} u_1 &= t^2 & u_2 &= t^{-2} & w_1 &= t^2(1 - t^{-6}) & w_2 &= t^2(1 - t^{-4})(1 - t^{-6}) \\ w_3 &= (1 - t^{-4})(1 + t^{-2}) & w_4 &= t^2(1 - t^{-2})(1 - t^{-4})(1 - t^{-6}) \\ p_1 &= -t^{-1} & p_2 &= t^{-4} & p_3 &= -t^{-3} & p_4 &= -t^{-7} \\ v_1 &= (1 + t^{-2})[(1 - t^{-2})(1 - t^{-6})]^{1/2} \\ v_2 &= t^{-2}(1 - t^{-6}) & v_3 &= -t(1 - t^{-4})(1 - t^{-6}) \\ u'_1 &= t^{-2} & u'_2 &= t^2 & w'_1 &= -t^4(1 - t^{-6}) & w'_2 &= t^8(1 - t^{-4})(1 - t^{-6}) \\ w'_3 &= -t^6(1 - t^{-4})(1 - t^{-2}) & w'_4 &= -t^{10}(1 - t^{-6})(1 - t^{-4})(1 - t^{-2}) \\ p'_1 &= -t & p'_2 &= t^4 & p'_3 &= -t^3 & p'_4 &= -t^7 \\ v'_1 &= -t^6(1 + t^{-2})[(1 - t^{-2})(1 - t^{-6})]^{1/2} \\ v'_2 &= -t(1 - t^{-6}) & v'_3 &= -t^9(1 - t^{-6})(1 - t^{-4}) \end{aligned} \tag{2.20}$$

where the trace cross-channel unitarity identity

$$\begin{aligned}
 &u_1^{-1}w_1t^{-1} + u_1^{-1}w_2t + u_1^{-1}w_4t^3 + u_2^{-1}w_3t + u_2^{-1}w_2t^3 + u_2^{-1}w_1t^3 + u_2w_1't^{-3} \\
 &\quad + u_2w_2't^{-3} + u_1w_4't^{-3} + u_2w_3't^{-1} + u_1w_2't^{-1} + u_1w_1't \\
 &\quad + w_3w_1't^{-1} + w_2w_1't + w_1w_2't^{-1} + w_1w_3't = 0
 \end{aligned} \tag{2.21}$$

has been used.

It can be checked directly that (2.20) satisfies (1.3), (1.4) and (2.21). The skein relation for T has the form

$$\begin{aligned}
 T^{-4} &= (q^{-1} - q^2 + q^4 - q^5)T^{-3} + q(1 - q^2 + q^3 + q^5 - q^6 + q^8)T^{-2} \\
 &\quad + q^5(-1 + q - q^3 + q^6)T^{-1} - q^{10}
 \end{aligned} \tag{2.22}$$

or

$$\begin{aligned}
 g^4 &= (1 - q^3 + q^5 - q^6)g^3 \\
 &\quad + q^3(1 - q^2 + q^3 + q^5 - q^6 + q^8)g^2 + q^8(-1 + q - q^3 + q^6)g - q^{14}.
 \end{aligned} \tag{2.23}$$

Considering

$$\begin{aligned}
 \bigcirc &= 2u_1 + 2u_2 + 2w_1t^2 + 2w_2t^4 + w_3t^2 + w_4t^6 \\
 &= t^5(t^3 + t + t^{-1} + t^{-3}) = t^5\bigcirc
 \end{aligned} \tag{2.24}$$

and

$$P_{3+} + \alpha P_{2+} + \beta P_{+} + \gamma P_{-} = \sigma P_0 \tag{2.25}$$

the coefficients in (2.25) can be determined with the help of (2.22)

$$\begin{aligned}
 \alpha &= -t^3(1 - t^6 + t^{10} - t^{12}) & \beta &= -t^{12}(1 - t^4 + t^6 + t^{10} - t^{12} + t^{16}) \\
 \gamma &= t^{40} & \sigma &= t^{25}(-1 + t^2 - t^6 + t^{12})
 \end{aligned} \tag{2.26a}$$

and the polynomial has the form

$$\begin{aligned}
 P_{3+} &= q^{3/2}(1 - q^3 + q^5 - q^6)P_{2+} + q^6(1 - q^2 + q^3 + q^5 - q^6 + q^8)P_{+} \\
 &\quad + q^{25/2}(-1 + q - q^3 + q^6)P_0 - q^{20}P_{-}.
 \end{aligned} \tag{2.26b}$$

This is the result of AW [7, 10].

Now let us use the extended state model to give a representation, skein relation and polynomial for $N=5$ that have not yet been derived by AW. Adding all the annihilation diagrams to the original diagram we have, for $N=5$,

$$\begin{aligned}
 \begin{array}{c} a \\ \diagdown \\ c \end{array} \begin{array}{c} b \\ \diagup \\ d \end{array} &= u \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} a \\ \diagup \\ a \end{array} + w \begin{array}{c} a \\ \diagdown \\ a \end{array} \begin{array}{c} < b \\ \diagup \\ a \end{array} + p \begin{array}{c} a \neq b \\ \diagdown \\ b \end{array} \begin{array}{c} b \\ \diagup \\ a \end{array} + v_1 \left(\begin{array}{c} 2 & 2 & 0 & 4 & -2 & -2 & -4 & 0 \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \\ 0 & 4 & 2 & 2 & -4 & 0 & -2 & -2 \end{array} \right) \\
 &+ v_2 \left(\begin{array}{c} 2 & 0 & -2 & 4 & 0 & -2 & -4 & 2 \\ \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \\ -2 & 4 & 2 & 0 & -4 & 2 & 0 & -2 \end{array} \right) + v_3 \left(\begin{array}{c} 0 & 2 & -2 & 4 \\ \diagdown & \diagup & \diagdown & \diagup \\ -2 & 4 & 0 & 2 \end{array} \right) + v_3 \left(\begin{array}{c} -2 & 0 & -4 & 2 \\ \diagdown & \diagup & \diagdown & \diagup \\ -4 & 2 & -2 & 0 \end{array} \right) \\
 &+ v_4 \left(\begin{array}{c} 2 & -2 & -4 & 4 \\ \diagdown & \diagup & \diagdown & \diagup \\ -4 & 4 & 2 & -2 \end{array} \right) + v_5 \left(\begin{array}{c} 0 & 0 & -2 & 2 \\ \diagdown & \diagup & \diagdown & \diagup \\ -2 & 2 & 0 & 0 \end{array} \right) + v_6 \left(\begin{array}{c} 0 & 0 & -4 & 4 \\ \diagdown & \diagup & \diagdown & \diagup \\ -4 & 4 & 0 & 0 \end{array} \right) \\
 &+ v_7 \left(\begin{array}{c} -2 & 2 & -4 & 4 \\ \diagdown & \diagup & \diagdown & \diagup \\ -4 & 4 & -2 & 2 \end{array} \right)
 \end{aligned} \tag{2.27}$$

$$\begin{aligned}
 \begin{array}{c} a \\ \diagdown \\ c \end{array} \begin{array}{c} b \\ \diagup \\ d \end{array} &= u' \begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ a \end{array} + w' \begin{array}{c} a \\ \diagdown \\ \text{---} \\ \diagup \\ b \end{array} + p \begin{array}{c} a \neq b \\ \diagdown \\ b \\ \diagup \\ a \end{array} + v'_1 \left(\begin{array}{c} 2 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ 0 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 0 \\ \diagup \\ 2 \end{array} + \begin{array}{c} -2 \\ \diagdown \\ 0 \end{array} \begin{array}{c} -2 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 0 \\ \diagdown \\ -2 \end{array} \begin{array}{c} -4 \\ \diagup \\ -2 \end{array} \right) \\
 &+ v'_2 \left(\begin{array}{c} 0 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ -2 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ 0 \end{array} \begin{array}{c} -2 \\ \diagup \\ 2 \end{array} + \begin{array}{c} -2 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 0 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 2 \\ \diagdown \\ -2 \end{array} \begin{array}{c} -4 \\ \diagup \\ 0 \end{array} \right) + v'_3 \left(\begin{array}{c} 2 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 0 \\ \diagup \\ -2 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ 2 \end{array} \begin{array}{c} -2 \\ \diagup \\ 0 \end{array} \right) + v'_3 \left(\begin{array}{c} 0 \\ \diagdown \\ 2 \end{array} \begin{array}{c} -2 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 2 \\ \diagdown \\ 0 \end{array} \begin{array}{c} -4 \\ \diagup \\ -2 \end{array} \right) \\
 &+ v'_4 \left(\begin{array}{c} -2 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 2 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ -2 \end{array} \begin{array}{c} -4 \\ \diagup \\ 2 \end{array} \right) + v'_5 \left(\begin{array}{c} 0 \\ \diagdown \\ 2 \end{array} \begin{array}{c} 0 \\ \diagup \\ -2 \end{array} + \begin{array}{c} 2 \\ \diagdown \\ 0 \end{array} \begin{array}{c} -2 \\ \diagup \\ 0 \end{array} \right) + v'_6 \left(\begin{array}{c} 0 \\ \diagdown \\ 4 \end{array} \begin{array}{c} 0 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ 0 \end{array} \begin{array}{c} -4 \\ \diagup \\ 0 \end{array} \right) \\
 &+ v'_7 \left(\begin{array}{c} 2 \\ \diagdown \\ 4 \end{array} \begin{array}{c} -2 \\ \diagup \\ -4 \end{array} + \begin{array}{c} 4 \\ \diagdown \\ 2 \end{array} \begin{array}{c} -4 \\ \diagup \\ -2 \end{array} \right). \tag{2.28}
 \end{aligned}$$

Substituting (2.27) and (2.28) into (1.3) it follows by the unitarity condition that

$$\begin{aligned}
 u'_i &= u_i^{-1} \quad i = 1, 2, 3 & p'_i &= p_i^{-1} \quad i = 1, 2, \dots, 6 \\
 w'_1 &= -p_1^{-2} w_1 & w'_2 &= p_2^{-2} (u_2^{-1} v_1^2 - w_2) & w'_3 &= -p_3^{-2} w_3 \\
 w'_4 &= p_4^{-2} (2p_3^{-1} v_2 v_3 - p_3^{-2} w_3 v_2^2 - w_4) & w'_5 &= p_5^{-2} (u_3^{-1} v_5^2 - w_5) \\
 w'_6 &= p_6^{-2} (2p_5^{-1} v_4 v_7 - u_3^{-1} v_6^2 - u_3^{-1} p_5^{-2} v_4^2 v_5^2 - 2u_3^{-1} p_5^{-1} v_4 v_5 v_6 - p_5^{-2} w_5 v_4^2 - w_6) \\
 v'_1 &= -u_2^{-1} p_2^{-1} v_1 & v'_2 &= -p_3^{-1} p_4^{-1} v_2 & v'_3 &= p_3^{-1} p_4^{-1} (u_3^{-1} w_3 v_2 - v_3) \\
 v'_4 &= -p_5^{-1} p_6^{-1} v_4 & v'_5 &= -u_3^{-1} p_5^{-1} v_5 & v'_6 &= u_3^{-1} p_6^{-1} (p_5^{-1} v_4 v_5 - v_6) \\
 v'_7 &= p_5^{-1} p_6^{-1} (p_5^{-1} w_5 v_4 - u_3^{-1} v_5 v_6 - u_3^{-1} p_5^{-1} v_4 v_5^2 - v_7).
 \end{aligned} \tag{2.29}$$

Substituting (2.27) and (2.28) into (1.4) and (1.12) and making use of the diagrammatic symbol expansion, rather than the detail of the tedious calculations, we give the equations that must be satisfied by the parameters appearing in (2.27) and (2.28), namely

$$\begin{aligned}
 u_1 &= t^2 & u_2 &= t^{-4} & u_3 &= t^{-6} & w_1 &= t^2(1-t^{-8}) \\
 w_2 &= t^2(1-t^{-6})(1-t^{-8}) & w_3 &= t^{-2}(1+t^{-2})(1-t^{-6}) \\
 w_4 &= t^2(1-t^{-4})(1-t^{-6})(1-t^{-8}) & w_5 &= (1+t^{-2})(1-t^{-6}) \\
 w_6 &= t^2(1-t^{-2})(1-t^{-4})(1-t^{-6})(1-t^{-8}) \\
 p_1 &= -t^{-2} & p_2 &= t^{-6} & p_3 &= -t^{-6} \\
 p_4 &= -t^{-10} & p_5 &= t^{-8} & p_6 &= t^{-14} \\
 v_1 &= t^{-1} [(1+t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} \\
 v_2 &= -t^{-4} [(1-t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} \\
 v_3 &= (1-t^{-6}) [(1-t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} & v_4 &= t^{-7}(1-t^{-8}) \\
 v_5 &= t^{-3}(1+t^{-2})(1-t^{-6}) & v_6 &= t^{-2}(1-t^{-6})(1-t^{-8}) \\
 v_7 &= t(1-t^{-4})(1-t^{-6})(1-t^{-8}).
 \end{aligned} \tag{2.30}$$

Combining (2.30) with (2.29), we derive the solution set

$$\begin{aligned}
 u'_1 &= t^{-2} & u'_2 &= t^4 & u'_3 &= t^6 & w'_1 &= -t^6(1-t^{-8}) \\
 w'_2 &= t^{12}(1-t^{-6})(1-t^{-8}) & w'_3 &= -t^{10}(1+t^{-2})(1-t^{-6}) \\
 w'_4 &= -t^{16}(1-t^{-4})(1-t^{-6})(1-t^{-8}) & w'_5 &= t^{14}(1+t^{-2})(1-t^{-6})^2 \\
 w'_6 &= t^{18}(1-t^{-2})(1-t^{-4})(1-t^{-6})(1-t^{-8}) \\
 p'_1 &= -t^2 & p'_2 &= t^6 & p'_3 &= -t^6 & p'_4 &= -t^{10} & p'_5 &= t^8 & p'_6 &= t^{14} \\
 v'_1 &= -t^9[(1+t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} \\
 v'_2 &= -t^{12}[(1+t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} \\
 v'_3 &= t^{14}(1-t^{-6})[(1+t^{-2})(1-t^{-6})(1-t^{-8})]^{1/2} & v'_4 &= -t^{15}(1-t^{-8}) \\
 v'_5 &= -t^{11}(1+t^{-2})(1-t^{-6}) & v'_6 &= t^{16}(1-t^{-6})(1-t^{-8}) \\
 v'_7 &= -t^{17}(1-t^{-4})(1-t^{-6})(1-t^{-8}).
 \end{aligned} \tag{2.31}$$

Here we give the trace cross-channel unitarity equation for checking purposes:

$$\begin{aligned}
 u_1(w'_1 t^2 + w'_2 + w'_4 t^{-2} + w'_6 t^{-4}) + u(w'_1 t^{-4} + w'_3 + w'_4 t^{-4} + w'_5 t^{-2}) \\
 + u_3(w'_2 t^{-4} + w'_3 t^{-2}) + u'_1(w_1 t^{-2} + w_2 + w_4 t^2 + w_6 t^4) \\
 + u'_2(w_1 t^4 + w_3 + w_4 t^4 + w_5 t^2) + u'_3(w_2 t^4 + w_3 t^2) \\
 + w_1 w'_3 t^2 + w_3 w'_1 t^{-2} + w_1 w'_4 t^{-2} + w_4 w'_1 t^2 + w_1 w'_5 \\
 + w_5 w'_1 + w_2 w'_3 t^2 + w_3 w'_2 t^{-2} + w_2 w'_2 + w_3 w'_3 = 0.
 \end{aligned} \tag{2.32}$$

Repeating the same procedure as before, the skein relation is given by

$$\begin{aligned}
 T^5 = t^2(1-t^{-8} + t^{-14} - t^{-18} + t^{-20})T^4 + t^{-4}(1-t^{-6} + t^{-10} - t^{-12} + t^{-14} - t^{-18} \\
 + t^{-20} + t^{-24} - t^{-26} + t^{-30})T^3 - t^{16}(1-t^{-4} + t^{-6} + t^{-10} - t^{-12} + t^{-16} - t^{-18} \\
 + t^{-20} - t^{-24} + t^{-30})T^2 - t^{-32}(1-t^{-2} + t^{-6} - t^{-12} + t^{-20})T + t^{-50}
 \end{aligned} \tag{2.33}$$



or for $g = qT^{-1}$ and $q = t^2$

$$(g - q^{10})(g - q^9)(g - q^7)(g + q^4)(g - 1) = 0. \tag{2.34}$$

The corresponding polynomial has the form

$$\begin{aligned}
 P_{4+} = q^2(1 - q^4 + q^7 - q^9 + q^{10})P_{3+} \\
 + q^8(1 - q^3 + q^5 - q^6 + q^7 - q^9 + q^{10} + q^{12} - q^{13} + q^{15})P_{2+} \\
 - q^{17}(1 - q^2 + q^3 + q^5 - q^6 + q^8 - q^9 + q^{10} - q^{12} + q^{15})P_+ \\
 + q^{40}P_- - q^{28}(1 - q + q^3 - q^6 + q^{10})P_0
 \end{aligned} \tag{2.35}$$

where P_{4+} represents the polynomial with four crossings. After calculation, some elementary link polynomials for spin 2 can be given:

Kauffman	extended
	$q^2(1 + q^5 + q^{10} + q^{15} + q^{20})$
	$q^4(1 + q^5 - q^9 + q^{10} - q^{13} - q^{14} + q^{15} - q^{18} - q^{19} + 2q^{20} - q^{24} + 2q^{25} - q^{28} - q^{29} + q^{30})$

and $q \rightarrow q^{-1}$ for the opposite orientation.

As was discussed above, if the extended state model is used then the skein relations and polynomials, in principle, can be calculated in a systematic way. Of course, the higher the spin, the more complicated the calculation. However, in the next section we shall give a general solution for constructing AW polynomials for arbitrary spin systems.

It is worth emphasising that our approach can never be regarded as a reinterpretation of AW new polynomials. It possesses much more power than this. In terms of this systematic scheme one is able to find a lot of new representations of the braid group and new skein relations. For instance, this aim has been arrived for the fundamental representations of the groups B_n , $C_n D_n$ and G_2 , and the six-dimensional representation of A_2 , the eight-dimensional representation of B_3 and so on [11].

3. Group approach and AW type polynomials

In this section we shall discuss the relationship between the Witten theory and AW type polynomials.

The physical understanding of knot polynomials was recently initiated by Witten [3, 9]. The theory is related to the decomposition of $R \times R$ where R is the fundamental representation of the group $SU(N)$

$$R \times R = \sum_{i=1}^s E_i.$$

In [3] Witten built up a general sketch of the method for interpreting link polynomials in terms of physical field theory. On an oriented three-dimensional manifold M with a compact simple group G , the integral of Chern-Simons 3-form on M is chosen to be the action

$$Z = \int_M L = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{3.1}$$

The Feynman path integral of a given Wilson line is called the partition function of M with the given link L

$$Z(S^3; W(C_1) \dots W(C_r)) = \int e^{iL} DA \prod_{a=1}^r \exp\left(in_a \int_{C_a} A\right). \tag{3.2}$$

Consider, as in figure 1, a 3-manifold M which is the connected sum of two pieces, M_L and M_R , joined along a 2-sphere S^2 . There may be links in M_L or M_R , but, if so, they are assumed not to penetrate the joining sphere S^2 . If for every 3-manifold X , we denote the partition function by $Z(X)$, then we observe that

$$\begin{aligned} Z(M)Z(S^3) &= \int \exp\left(i \int_M L\right) DA(x) \prod_{a=1}^r \exp\left(in_a \int_{C_a} A_a(x)\right) \int \exp\left(i \int_{S^3} L\right) DA(y) \\ &= \int \exp\left(i \int_{M_1} L\right) DA(x_1) \prod_b \exp\left(in_b \int_{C_b \in M_1} A_b(x'_1)\right) \\ &\quad \times \int \exp\left(i \int_{M_2} L\right) DA(x_2) \\ &\quad \times \prod_a \exp\left(in_a \int_{C_a \in M_2} A(x'_2)\right) = Z(M_1)Z(M_2) \end{aligned} \tag{3.3}$$

where $x \in M, y \in S^2, x_1 \in M_1, x_2 \in M_2$. The identity (see figure 1) $M + S^3 = M_1 + M_2$ has been used: $Z(S^3)$ denotes the partition function of S^3 with no links. The formula (3.3) can be rewritten as

$$\frac{Z(M)}{Z(S^3)} = \frac{Z(M_1)}{Z(S^3)} \frac{Z(M_2)}{Z(S^3)} \tag{3.4}$$

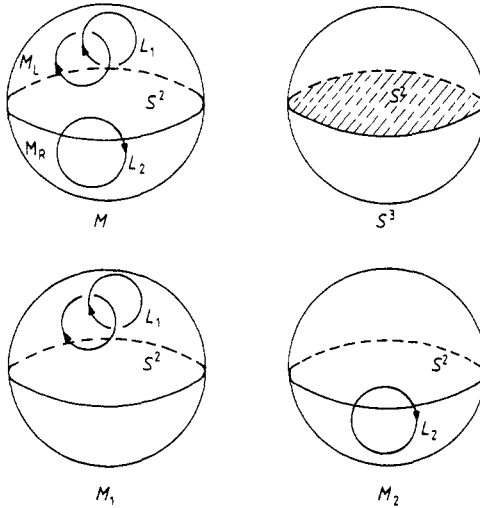


Figure 1. A 3-manifold M which is the connected sum of two pieces M_L and M_R , joined along a 2-sphere S^2 . Similarly, a 3-sphere S^3 can be cut along its ‘equator’, a S^2 sphere, being two 3-balls B_L and B_R . Cutting both M and S^3 as indicated above, assuming there are no link lines penetrating the 2-sphere S^2 , the pieces can be rearranged into M_1 and M_2 , i.e. $M + S^3 = M_1 + M_2$.

If we introduce the normalised expectation value of a link L , defined by $P(L) = Z(S^3; L) / Z(S^3)^\dagger$ then (3.4) becomes

$$P(C_1, C_2) = P(C_1)P(C_2) \tag{3.5}$$

for unknotted (unlinked) circles C_1 and C_2 . For example

$$P[\bigcirc \bigcirc] = P[\bigcirc]P[\bigcirc]. \tag{3.6}$$

Equation (3.5) can be generalised by cutting the manifold to separate the circles and repeatedly using (3.5) to the case of arbitrary collection of unknotted, unlinked Wilson lines on S^3 .

If there are link lines in M (means S^3) penetrating through S^2 , one will have a Riemann sphere (the joining sphere S^2) with marked points [3]. The Feynman path integral on M_R (with boundary S^2) determines a vector ψ in Hilbert space \mathcal{H} . The path integral on M_L (the boundary is the same but it has opposite orientation) determines a vector χ in $\tilde{\mathcal{H}}$ (canonically the dual of \mathcal{H}). It is easy to understand that the partition function on the connected sum M (perhaps with link line l) is a natural

$\dagger M$ is taken as S^3 .

pairing of the vectors χ, ψ , i.e.‡

$$P(M) = (\chi, \psi). \tag{3.7}$$

From the work of Verlinde [11], the dimension of physical Hilbert space for an arbitrary collection of marked points on S^2 can be determined from the knowledge of N_{ijk} . We are more interested in the case with four marked points (external charges) on the Riemann sphere S^2 . Consider that there are four charges (marked points p_1, p_2, p_3, p_4) on S^2 with representation R, R and \bar{R}, \bar{R} of G . The physical Hilbert space (at large k) is s -dimensional if the direct product of the irreducible representation R decomposes to s distinct irreducible representations of G

$$R \otimes R = \sum_{i=1}^n \oplus E_i. \tag{3.8}$$

We conventionally denote the irreducible representation whose highest weight is the twice of that of R by E_1 . Certainly for any $n + 1$ vectors in the n -dimensional Hilbert space, there would be a linear relation

$$\alpha\psi + \alpha_1\psi_1 + \dots + \alpha_n\psi_n = 0 \tag{3.9}$$

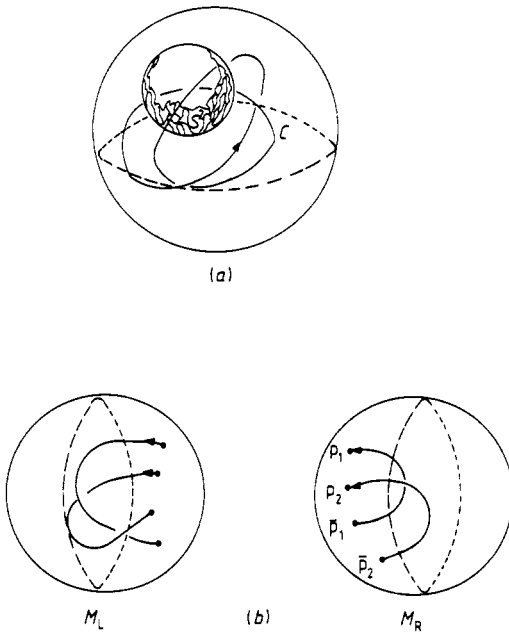


Figure 2. (a) M is a S in which curve C is embedded, if it were sketched as shown here, the boundary of that 'ball' would be considered as a single point. Cutting M around the intersection of C and a 2-sphere s , we will get two balls embedded with link lines (see (b)). (b) M_L and M_R are an 'interior' piece and an 'exterior' piece respectively. The boundary of the figure should not be considered as a single point.

As shown in figure 2(a), we cut into two pieces the 3-manifold by the two-dimensional Riemann sphere S^2 which intersects with the link line at points (p_1, p_2) and (\bar{p}_1, \bar{p}_2) . The exterior piece is denoted by M_L ; the interior piece is denoted by

‡ Normalised partition function.

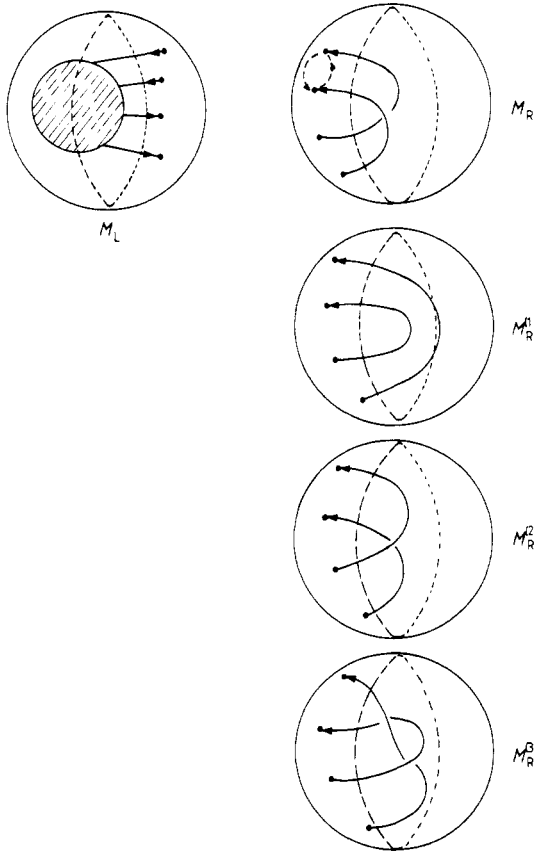


Figure 3. Reduction relations and the surgery for three eigenvalues. For more eigenvalues the extensions are obvious.

M_R (see, for example, figure 2(b), more generally see figure 3). The Hilbert spaces associated with the boundary of M_L and that of M_R are \mathcal{H}^- and \mathcal{H} respectively. Making a deformation such that on the boundary of M_R the marked point p_1 goes to the original position of p_2 , and point p_2 goes to the original position of p_1 . This changes M_R into $M_R^{(1)}$. Gluing it to M_L , we have another link C' which is the same as C outside of S^2 but different inside. This requires studying the Hilbert space which arises as the space of conformal blocks for the p_1, p_2, p_3, p_4 four-point correlation function on S^2 .

In conformal field theory the correlation function satisfies [12]

$$\left((C_g + k) \frac{\partial}{\partial Z_i} - \sum_{j \neq i} \frac{t_i^a t_j^a}{Z_j - Z_i} \right) \Phi_{R, \mu_1}(Z_1) \Phi_{R, \mu_2}(Z_2) \Phi_{\bar{R}, \mu_3}(Z_3) \Phi_{\bar{R}, \mu_4}(Z_4) = 0 \tag{3.10}$$

here k is the centre charge of Kac-Moody algebra. The C_g is determined by the group G due to $f^{acd} f^{bcd} = C_g^{ab}$ [12], where f^{abc} is the structure constant of G . The change of configuration under the deformation can be regarded as a result of the half monodromy operation. That is, the operator B in [9]. The eigenvalues of B are shown in [9], they are

$$\lambda_i = \pm \exp(i\pi(2\Delta_R - \Delta_{E_i})) \tag{3.11}$$

where the sign + or - corresponds to whether E_i appears symmetrically or antisymmetrically in $R \otimes R$; Δ_R or Δ_{E_i} is the conformal weight of the primary field [12] transforming as R or E_i . It is given generally in [12] from the Wess-Zumino chiral model by

$$\Delta_R = \frac{C_R}{C_g + k} \quad \Delta_{E_i} = \frac{C_{E_i}}{C_g + k}. \tag{3.12}$$

Following [3, 7, 9], the characteristic equation of the operator B is

$$\prod_{i=1}^n (B - \lambda_i) = B^n - \left(\sum_{i=1}^n \lambda_i \right) B^{n-1} + \dots + (-1)^{n-1} \left(\prod_{i=1}^n \lambda_i \right) \\ \times \sum_{i=1}^n \lambda_i^{-1} B + (-1)^n \left(\prod_{i=1}^n \lambda_i \right) I = 0 \tag{3.13}$$

where λ_i are eigenvalues of B . The reliability of the multi-eigenvalue extension for operator B will be confirmed later. Correspondingly the dependent $(n + 1)$ vectors in Hilbert space H_R associated with boundary M_R obey

$$(B^n \psi) - \left(\sum_{i=1}^n \lambda_i \right) (B^{n-1} \psi) + \left(\sum_{i < j}^n \lambda_i \lambda_j \right) (B^{n-2} \psi) + \dots \\ + (-1)^{n-1} \left(\prod_{i=1}^n \lambda_i \right) \left(\sum_{i=1}^n \lambda_i^{-1} \right) (B \psi) + (-1)^n \left(\prod_{i=1}^n \lambda_i \right) \psi = 0. \tag{3.14}$$

Referring to [3, 9, 12], it is easy to determine λ_i for $SU(2)$. Let Δ_R (Δ_{E_i}) be the conformal weight of the primary conformal field [3, 9, 12] transformed as $R^{(j)}$ (E_i), the eigenvalues of B are [13]

$$\lambda_i = \pm \exp\{i\pi(2\Delta_R - \Delta_{E_i})\} \tag{3.15}$$

where the sign corresponds to whether E_i appears symmetrically or antisymmetrically in λ_i . In the present situation it depends on the level of E_i . For the j representation of $SU(2)$ it is very easy to see that Δ is related to the Casimir corresponding to a weight j [12]

$$\Delta^{(j)} = j(j + 1)/(k + 2). \tag{3.16}$$

Now let us consider the skein relations and the corresponding polynomials for different spins j .

(i) $j = 1$. From (3.14) and (3.15) it follows that

$$\Delta_R = \Delta^{(1)} = 2/(k + 2) \quad \Delta_{E_1} = \Delta^{(0)} = 0 \quad \Delta_{E_2} = \Delta^{(1)} = 2/(k + 2) \\ \Delta_{E_3} = \Delta^{(2)} = 6/(k + 2) \quad \lambda_1 = q^2 \quad \lambda_2 = -q \quad \lambda_3 = q^{-1} \\ q = \exp(2i\pi/k + 2) \quad (B - q^2)(B + q)(B - q^{-1}) = 0$$

which agrees with (2.14) with $B = T^{-1}q^{-1}$ and $q = t$. Equation (3.14) becomes

$$B^3 \psi - q^{-1}(q^3 - q^2 + 1)B^2 \psi - q^{-2}(q^5 - q^3 + q^2)B \psi + q^2 \psi = 0.$$

Taking the natural pairing of χ in H_L with (3.14), which implies that M_2 is glued with M_1 and its diffeomorphism M'_1, M'_2, M'' , respectively, one gets

$$\langle \chi B^3 \psi \rangle - q^{-1}(q^3 - q^2 + 1)\langle \chi B^2 \psi \rangle - q^{-2}(q^5 - q^3 + q^2)\langle \chi B \psi \rangle - q^2\langle \chi \psi \rangle = 0.$$

Taking account of the correction factor of framing (see [3]) $f = \exp(-i2\pi A_R)$ we have

in general

$$\langle \chi \psi \rangle = Z(L_{-1}) \quad \langle \chi B^{m+1} \psi \rangle = f^{m+1} Z(L_{+m}) \quad m = 0, 1, 2, \dots \quad (3.17)$$

The power indices in (3.17) indicate the path of regulation, i.e. the ‘auxiliary’ lines twist the curve in $M^3(m+1)$ times. For $j = 1$ one has $f = q^{-2}$ the (cubic) skein relation (3.11) has the form

$$Z(L_{+2}) - q(q^3 - q^2 + 1)Z(L_{+1}) - q^2(q^5 - q^3 + q^2)Z(L_0) + q^8 Z(L_{-1}) = 0.$$

It coincides exactly with AW for $n = 2$ or $N = 3$.

(ii) $j = \frac{3}{2}$. In the case that H_R is four dimensional we have

$$\lambda_1 = -q^{15/4} \quad \lambda_2 = q^{11/4} \quad \lambda_3 = -q^{3/4} \quad \lambda_4 = q^{-9/4} \quad f = q^{-15/4}.$$

Correspondingly

$$(B + q^{15/4})(B - q^{11/4})(B + q^{3/4})(B - q^{-9/4}) = 0$$

which agrees with AW for $B = q^{-9/4} T^{-1}$. The polynomial can be given by

$$Z(L_{+3}) + q^{3/2}(q^6 - q^5 + q^3 - 1)Z(L_{+2}) - q^6(q^8 - q^6 + q^5 + q^3 - q^2 + 1)Z(L_{+1}) \\ - q^{25/2}(q^6 - q^3 + q - 1)Z(L_0) + q^{20}Z(L_{-1}) = 0.$$

This is the AW polynomial for $N = 4$.

(iii) $j = 2$.

$$\lambda_1 = q^6 \quad \lambda_2 = -q^5 \quad \lambda_3 = q^3 \quad \lambda_4 = -1 \quad \lambda_5 = q^{-4} \quad f = q^{-6}$$

$$Z(L_{+4}) - q^2(q^{10} - q^9 + q^7 - q^4 + 1)Z(L_{+3}) \\ - q^8(q^{15} - q^{13} + q^{12} + q^{10} - q^9 + q^7 - q^6 + q^5 - q^3 + 1)Z(L_{+2}) \\ + q^{17}(q^{15} - q^{12} + q^{10} - q^9 + q^8 - q^6 + q^5 + q^3 - q^2 + 1)Z(L_{+1}) \\ + q^{24}(q^{14} - q^{10} + q^7 - q^5 + q^4)Z(L_0) - q^{40}Z(L_{-1}) = 0. \quad (3.18)$$

This equation coincides with (2.35) if we identify Z with P .

(iv) $J = \frac{5}{2}$.

$$\lambda_1 = -q^{35/4} \quad \lambda_2 = q^{31/4} \quad \lambda_3 = -q^{23/4} \quad \lambda_4 = q^{11/4} \\ \lambda_5 = -q^{-5/4} \quad \lambda_6 = q^{-25/4} \quad f = q^{-35/4}.$$

The skein relation for the polynomial is

$$Z(L_{+5}) + q^{5/2}(q^{15} - q^{14} + q^{12} - q^9 + q^5 - 1)Z(L_{+4}) \\ + q^{10}(-q^{24} + q^{22} - q^{21} - q^{19} + q^{18} - q^{16} + q^{15} \\ - q^{14} + q^{12} - q^{10} - q^7 + q^4 - 1)Z(L_{+3}) \\ - q^{43/2}(q^{27} - q^{24} + q^{22} - q^{21} + q^{20} - q^{18} + q^{17} - q^{14} + q^{13} - q^{10} \\ + q^9 - q^7 + q^6 - q^5 + q^3 - 1)Z(L_{+2}) \\ + q^{36/2}(q^{24} - q^{20} + q^{17} + q^{14} - q^{12} + q^{10} \\ - q^9 + q^8 - q^6 + q^5 + q^3 - q^2 + 1)Z(L_{+1}) \\ + q^{105/2}(q^{15} - q^{10} + q^6 - q^3 + q - 1)Z(L_0) \\ - q^{70}Z(L_{-1}) = 0. \quad (3.19)$$

This is a sixth-power skein relation, which is a new one, i.e. it has not been derived by either AW or the extended diagrammatic scheme. We believe that one can obtain this relation from a $6^2 \times 6^2$ representation of the braid group. It is interesting to note that (3.18) exactly coincides with (2.35) except for the lower spin states.

The non-trivial nature of the coincidence between Witten's approach and the extended diagrammatic calculations is in the framing factor f defined by (3.8). Now let us make a general discussion.

The advantage of Witten's approach is that it enables one to immediately write down the eigenvalues which are related to the 1+1 CFT. For $n\lambda_i$ we have

$$\prod_{i=1}^n (B - \lambda_i) = 0 \tag{3.20}$$

where

$$\lambda_i = \pm q^{(C_R - 1/2 C_{E_i})} \tag{3.21}$$

and

$$q = \exp[2i\pi / (C_g + k)].$$

The difference between B and the braid group representation (BGR) is only in a normalisation factor. We can thus regard B as the T (BGR) up to a factor. It is very easy to construct the skein relations in terms of Witten's approach if the framing factor is shown to be universal, since (3.20) and (3.21) are correct for any Lie algebras (see [5, 14]). In this case, because

$$T^n \psi + \alpha_1 T^{n-1} \psi + \dots + \alpha_{n-1} T \psi + \alpha_n \psi = 0 \tag{3.22}$$

where

$$\alpha_1 = -\sum_i \lambda_i, \dots, \alpha_{n-1} = \alpha_n \sum_i (-\lambda_i^{-1}) \quad \alpha_n = \sum_{i=1}^n (-\lambda_i)$$

and the framing factor f defined by

$$(\chi, T^m \psi) = f^m P(L_{(m-1)}) \tag{3.23}$$

with

$$f = \exp(-2i\pi \Delta_R) \tag{3.24}$$

we use χ to act on the LHS of (3.22) and consider (3.23) to give the skein relation

$$P(L_{n-1}) + f^{-1} \alpha_1 P(L_{n-2}) + \dots + f^{-(n-1)} \alpha_{n-1} P(L_0) + f^{-n} \alpha_n P(L_{-1}) = 0. \tag{3.25}$$

Now let us show that the framing factor f given by (3.23) works for any Lie algebras. The basic reason consists in the Markov trace.

As shown in [7, 14], the Markov trace is given by

$$\Phi(A) = \text{Tr}(AH) \tag{3.26}$$

where A represents a braiding block and H is the tensor product of diagonal matrices h , namely

$$H = h \otimes h \otimes \dots \otimes h.$$

For the Lie algebras (with multiplicity one) Reshetikhin [14] gave the general form of h

$$h_{ab} = \delta_{ab} t^{-2\langle \delta, w_a \rangle} = \delta_{ab} t^{-L(a)} \tag{3.27}$$

where δ is the half sum of the simple roots and W_a denotes the weight labelled by index a . The point of the Markov trace is that τ and $\bar{\tau}$ are independent of index a where

$$\sum_b T_{ab}^{ab} h_{ab} = \tau \quad \sum_b (T^{-1})_{ab}^{ab} h_{bb} = \bar{\tau}. \tag{3.28}$$

In the AW approach for any characteristic equation (3.22) the skein relation can be formed by taking the Markov trace first

$$\text{Tr}(T^{n-1}H) + \alpha_1 \text{Tr}(T^{n-2}H) + \dots + \alpha_{n-1} \text{Tr}(H) + \alpha_n \text{Tr}(T^{-1}H) = 0$$

then multiplying each term by the corresponding power of the factor $(\bar{\tau}/\tau)^{e(A)/2}$. It gives the result

$$P(L_{n-1}) + \alpha^{1/2} \alpha_1 P(L_{n-2}) + \dots + \alpha^{-(1/2)(n-1)} \alpha_{n-1} P(L_0) + \alpha^{-(1/2)n} \alpha_n P(L_{-1}) = 0 \tag{3.29}$$

where $e(A)$ is given in [7] and

$$\alpha = \bar{\tau}/\tau. \tag{3.30}$$

Since the index a_{\max} corresponds to the weight W_R , we choose the element

$$T_{aa}^{aa}|_{a=a_{\max}} = t^{-(W_R)^2} \tag{3.31}$$

so that

$$\tau = t^{-(W_R^2 + W_R 2\delta)} = t^{-C_R}. \tag{3.32}$$

We thus obtain the relationship between the framing factor f in Witten's approach and α in the AW Markov trace based on the general discussions of Reshetikhin [3, 7, 14]:

$$\alpha = f^{-1} \quad \bar{\tau} = \tau^{-1}. \tag{3.33}$$

The equality

$$q = t \tag{3.34}$$

where t relates to the unknotted contribution through

$$[\bigcirc] = \sum_a t^{-2(\delta, W_a)} \tag{3.35}$$

is due to the trace cross-channel unitarity. The verification is as follows.

There are two possibilities for closure of the graph



One of them is trivial



The other provides the trace cross-channel unitarity

$$\begin{aligned} \text{Diagram} &= \sum_{a,b,c} S_{ca}^{ca} (S^{-1})_{cb}^{cb} t^{-L(a)} t^{-L(b)} t^{-L(c)} \\ &= \sum_c \left(\sum_a S_{ca}^{ca} t^{-L(a)} \right) \left(\sum_b (S^{-1})_{cb}^{cb} t^{L(b)} \right) t^{-L(c)} \\ &= \sum_c t^{-L(c)} = [\bigcirc] \end{aligned}$$

namely, the trace cross-channel unitarity [1, 2, 10] is automatically satisfied provided (3.34) holds.

For many cases the resultant decomposition of $R \times R$ contains such components as $C_{E_i} = 0$ (one-dimensional representation). Under such a case the ‘first’ element of T can be normalised to be 1, i.e.

$$\bar{T}_{aa}^{aa}|_{a=a_{\max}} = 1.$$

By virtue of the form of (3.22) the above discussions are still valid. We thus conclude that Witten’s approach can be extended to the cases of Lie algebras which include the spin models.

Witten’s approach provides a general scheme for constructing link polynomials based on the Casimirs (Δ_R and Δ_{E_i}) of classical Lie algebras. If we restrict ourselves only to the link polynomials no BGR is needed. Conversely, by virtue of [3, 14], we can regard the traces of the submatrices and the Markov trace for T as the constraint conditions on the parameters in T . In this way the calculations to derive the BGR can be simplified. In the next section we take the case of $\text{spin } \frac{5}{2}$ as an example to compute the associated BGR with the simplification. It is worth noting that cubic or higher powered reduction relations can no longer be determined by only the reduction relation as in the case of Jones (see (3.6) and [3, 14]). However, in these cases (3.35) is the universal expression for $[\square]$.

4. The direct calculations for the case of $\text{spin } \frac{5}{2}$

The Casimir eigenvalues for $SU(2)$ are simply $j(j+1)$ so that the eigenvalues of BGR under the case are

$$1 \quad -t^5 \quad t^9 \quad -t^{12} \quad t^{14} \quad -t^{15} \tag{4.1}$$

where

$$t = \exp(2\pi i/(k+2)).$$

The general considerations lead to the block-diagonal form for the BGR T [7, 8]

$$T = (A^{(5)}, \dots, A^{(1)}, A^{(0)}, A^{(1)}, \dots, A^{(5)}) \tag{4.2}$$

where

$$\begin{aligned}
 A^{(5)} &= 1 & A^{(4)} &= \begin{bmatrix} 0 & -t^{5/2} \\ -t^{5/2} & 1-t^5 \end{bmatrix} \\
 A^{(3)} &= \begin{bmatrix} 0 & 0 & t^5 \\ 0 & t^4 & q_1 \\ t^5 & q_1 & (1-t^4)(1-t^5) \end{bmatrix} \\
 A^{(2)} &= \begin{bmatrix} 0 & 0 & 0 & -t^{15/2} \\ 0 & 0 & -t^{11/2} & q_1 \\ 0 & -t^{11/2} & (1-t^4)(t^3+t^4) & q_3 \\ -t^{15/2} & q_2 & q_3 & (1-t^3)(1-t^4)(1-t^5) \end{bmatrix}
 \end{aligned}$$

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & t^{10} & & \\ 0 & 0 & 0 & t^7 & q_4 & & \\ 0 & 0 & t^6 & q_5 & q_6 & & \\ 0 & t^7 & q_5 & p_1 & q_7 & & \\ t^{10} & q_4 & q_6 & q_7 & (1-t^2)(1-t^3)(1-t^4)(1-t^5) & & \end{bmatrix}$$

$$A^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -t^{25/2} \\ 0 & 0 & 0 & 0 & -t^{17/2} & q_8 & q_9 \\ 0 & 0 & 0 & -t^{13/2} & q_9 & q_{10} & q_{11} \\ 0 & 0 & -t^{13/2} & p_2 & q_{11} & q_{12} & q_{13} \\ 0 & -t^{17/2} & q_9 & q_{11} & p_3 & q_{13} & \prod_{m=1}^5 (1-t^m) \\ -t^{25/2} & q_8 & q_{10} & q_{12} & q_{13} & \prod_{m=1}^5 (1-t^m) & \end{bmatrix}$$

where the parameters $q_1, \dots, q_{13}, p_1, \dots, p_3$ will be determined in the following.

(i) By taking the trace of $A^{(1)}$ and using the eigenvalues $1, \dots, t^{14}$ we have

$$p_1 = t^2 + t^3 + t^4 - t^5 - 2t^6 - 2t^7 - t^8 + t^9 + t^{10} + t^{11}.$$

(ii) By taking the Markov trace we get

$$(1-t^3)(1-t^4)(1-t^5)(t^{-5/2}) + p_1 t^{-3/2} + p_2 t^{-1/2} + t^6 t^{1/2} = t^{-5/2}$$

$$p_2 = t^4 + t^5 + t^6 - t^7 - t^8 - t^9.$$

(iii) With (4.1) the trace of $A^{(0)}$ leads to

$$p_3 = t + t^2 + 2t^8 + 3t^9 + t^{10} - t^4 - 3t^5 - 2t^6 - t^{12} - t^{13}.$$

The off-diagonal parameters q should be determined in terms of the Υ_{BE} , which can be performed by means of the extended Υ_B state model. The standard device is described in [10]. The results are as follows:

$$(q_1)^2 = t^{14} - t^{10} - 2t^9 - t^8 + t^{13} + t^5 + t^4$$

$$(q_2)^2 = (1-t^5)(t^{10} - t^{11}) + t^4 q_1^2$$

$$(q_3)^2 = (1-t^5)^2(1-t^4)(t^3 + t^4) + t^5(1-t^4)(1-t^5) - (1-t^4)^2(t^3 + t^4)^2(1-t^5) - t^{11}(1-t^4)(1-t^5)$$

$$(q_4)^2 = (1-t^5)(t^{15} - t^{14}) + (q_2)^2 t^4$$

$$(q_5)^2 t^4 = (1-t^4)^2(1+t)^2 t^{12} + (1-t^4)(t^3 + t^4)(t^{11} - t^{12})$$

$$(q_6)^2 = (1-t^4)^2(1-t^5)^2 t^6 + (1-t^4)(1-t^5)(t^{10} - t^{12}) + (q_1)^2(1-t^4)(t^3 + t^4) - q_5^2(1-t^5)$$

$$(q_7)^2 = p_1(1-t^5)^2 - p_1^2(1-t^5) + (1-t^3)(1-t^4)(1-t^5)(t^5 - t^{14}) - (q_5)^2(1-t^4)(1-t^5)$$

$$(q_8)^2 = (1-t^5)(t^{20} - t^{17}) + q_4^2 t^4$$

$$t^4(q_9)^2 = (1-t^4)(t^3 + t^4)(t^{14} - t^{13}) + t^6 q_5^2$$

$$(q_{10})^2 = (1-t^4)(1-t^5)(t^{15} - t^{13}) + t^6(q_3)^2 + (q_2)^2(1-t^4)(t^3 + t^4) - (q_9)^2(1-t^5)$$

$$(q_{11})^2 t^4 = p_2(1-t^4)^2(t^3 + t^4)^2 + p_1 t^{11} + t^6 q_3^2 - t^{13} p_1 - (p_2)^2(1-t^4)(t^3 + t^4)$$

$$(q_{12})^2 = p_2(1-t^4)^2(1-t^5)^2 + p_1q_1^2 + (1-t^3)(1-t^4)(1-t^5)(t^{10}-t^{13})$$

$$- p_2^2(1-t^5)(1-t^4) - q_{11}(1-t^5)$$

$$(q_{13})^2 = p_3(1-t^5)^2 + (1-t^2)(1-t^3)(1-t^4)(1-t^5)(t^5-t^{17})$$

$$- p_3^2(1-t^5) - q_9^2(1-t^3)(1-t^4)(1-t^5) - q_{11}^2(1-t^4)(1-t^5).$$

Thus all of the unknown parameters have been expressed in terms of t , e.g. we obtain the explicit form of the BGR for spin $\frac{5}{2}$.

Acknowledgments

We are grateful to Professor C N Yang for bringing to our attention the topics discussed in this paper and for many interesting comments. We thank Professors Y S Wu and H C Lee for many helpful discussions. We are obliged to Professor D Friedan who called our attention to Witten's work and Professor H J De Vega for helpful discussions. Much thanks to Professor M Wadati for helpful discussions.

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